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Shear dispersion along a rotating axle in a closely fitting shaft

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A formula is derived for the longitudinal shear dispersion coefficient of a solute in a laminar flow along and around a rotating cylindrical axle in an off-centre closely fitting shaft. The rotation drives a circulation which augments diffusive mixing around the axle and reduces the eventual rate of longitudinal spreading. A simple approximation is shown to give accurate results for the important special case of a cylindrical shaft.

1. Introduction

When an axle is inserted into a long closely fitting shaft it will almost inevitably be off centre. When fluid is pumped along the remaining non-uniform gap (for lubrication or cleaning) the longitudinal velocity will have an exaggerated non uniformity (Snyder & Goldstein 1965). This has dramatic consequences for the longitudinal spreading of any solute or small particles carried by the flow. Sankarasubramanian & Gill (1971) give an example where the eventual longitudinal dispersion coefficient is 250 times that for a concentric annulus.

In the context of bends in pipes, Erdogan & Chatwin (1967) and Johnson & Kamm (1986) have shown that centrifugally driven transverse flow augments the transverse mixing and can markedly reduce the eventual rate of longitudinal dispersion. The rotation of an axle in a stationary shaft will drive a transverse flow around the narrow gap. The purpose of the present work is to calculate the consequential reduction in the longitudinal dispersion coefficient, when the rotation is sufficiently slow that the solute is well-mixed across the gap.

At even higher rotation rates there can be homogenization round the azimuthal paths well before diffusion would be complete across the gap (Rhines & Young 1983; Pedley & Kamm 1988). Also, the helical flow can become unstable to annular Taylor vortices (Kaye & Elgar 1958). The present analysis does not extend to these rapid rotation regimes.

2. Two-dimensional equations for narrow-gap flows

When the solute (or small particles) first enters the annular gap, the mixing process is fully three-dimensional and comparatively inefficient (i.e. with molecular rates of diffusion). Then gradually there is mixing across the narrow gap between the axle and the shaft. So, the mixing process becomes two-dimensional with shear-augmented dispersion (Smith 1990). Further along the shaft the concentration becomes well-mixed around the axle and there is a dramatic growth in the effective rate of longitudinal dispersion (Muller & Bittleston 1991). Finally, the dispersion process is fully effective with longitudinal dispersion coefficient many orders of magnitude greater than molecular rates of diffusion and can be modelled by a one-dimensional diffusion equation (Taylor 1953; Sankarasubramanian & Gill 1971). Although we are concerned with the final regime, the magnitude of the shear dispersion coefficient depends upon the two-dimensional structure of the flow and mixing. So our calculations begin in the two-dimensional regime.

The flow geometry is illustrated in figure 1. We use cylindrical polar coordinates: θ measures the angle around the axle, and z measures the distance along the axle. The gap width $h(\theta)$ is assumed to be small relative to the mean radius a of the annular region, and we shall assume that the flow is well-mixed across the gap. If there is no z-variation in density, geometry or flow, then the two-dimensional equation for the conservation of mass is

$$\partial_\theta(hv) = 0, \tag{2.1}$$

where v is the velocity component around the axle (averaged across the velocity profile in the gap).

If the viscous drag at the boundaries is in local balance with the driving forces (i.e. steady flow with negligible advective flux of momentum around the axle), then for a laminar flow the momentum equations take the lubrication theory forms (Schlichting 1955, section 6c):

$$\frac{1}{a\rho}\partial_\theta p = -12\frac{\nu}{h^2}(v-\tfrac{1}{2}\Omega a), \tag{2.2a}$$

$$\frac{1}{\rho}\partial_z p = -12\frac{\nu}{h^2}w. \tag{2.2b}$$

Here $p(\theta, z)$ is the excess pressure (above hydrostatic), ρ the constant density, ν the constant kinematic viscosity, Ω the angular velocity of the axle rotation and $w(\theta)$ the axial velocity (averaged across the gap).

The two-dimensional shear dispersion equation is

$$h\,\partial_t c+\frac{hv}{a}\partial_\theta c+hw\,\partial_z c = \frac{1}{a^2}\partial_\theta(h[K+D_{\theta\theta}]\,\partial_\theta c)+\frac{1}{a}\partial_\theta(hD_{\theta z}\,\partial_z c)$$
$$+\frac{1}{a}\partial_z(hD_{z\theta}\,\partial_\theta c)+\partial_z[h(K+D_{zz})\,\partial_z c]. \tag{2.3}$$

Here c is the solute concentration, K the constant molecular diffusivity, and $D_{\theta\theta}, D_{\theta z}, D_{z\theta}, D_{zz}$ are the components of the two-dimensional shear dispersion tensor. Important general properties of the dispersion tensor are the symmetry

$$D_{\theta z} = D_{z\theta} \tag{2.4a}$$

and the positive definiteness (Smith 1990, equation (5.4a))

$$D_{\theta z}^2 < D_{\theta\theta}D_{zz}. \tag{2.4b}$$

For laminar flow with a quadratic velocity profile in the gap Smith (1990, equation (7.2)) gives the explicit formulae

$$D_{\theta\theta} = \frac{h^2}{210K}(v^2-v\Omega a+2\Omega^2a^2), \quad D_{\theta z} = D_{z\theta} = \frac{h^2 w}{210K}(v-\tfrac{1}{2}\Omega a), \quad D_{zz} = \frac{h^2w^2}{210K}. \tag{2.5a–c}$$

The numerical factors are simply the appropriate combinations of the dispersion coefficients for plane Poiseuille and plane Couette flow.

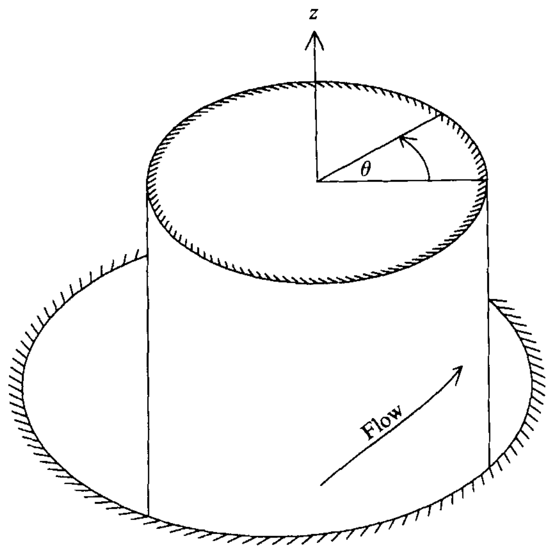

FIGURE 1. Sketch showing the non-uniform gap between a rotating axle and a stationary shaft.

The timescale for mixing across the gap can be estimated as

$$t_{\text{gap}} = \frac{h^2}{\pi^2 K}. \tag{2.6}$$

In this time the mean rotation speed $\frac{1}{2}\Omega a$ of the fluid in the gap will have carried material a distance of $\frac{1}{2}\Omega a t_{\text{gap}}$. This is less than the circumference $2\pi a$ (and thus the solute is well-mixed across the gap) if the angular velocity Ω satisfies the inequality

$$\frac{\Omega h^2}{K} < 4\pi^3 \tag{2.7}$$

For the momentum equations the corresponding inequality for the validity of lubrication theory involves ν instead of K. The absence of Taylor vortices (Kaye & Elgar 1958, section 6*c*) requires the stability condition

$$\frac{\Omega a^{\frac{1}{2}} h^{\frac{3}{2}}}{\nu} < 41.2. \tag{2.8}$$

So, unless h/a is large, the absence of Taylor vortices implies the appropriateness of lubrication theory.

In some applications the gap width is not particularly small. For rotating liners in boreholes (Arceneaux & Smith 1986, table 2) the average outer to inner radius ratio is 1.5. Fortunately, if $h(\theta)$ is interpreted suitably, then the coefficients in (2.1)–(2.5) are accurate to order $(h/a)^2$. If the radial positions of the inner and outer boundaries are

$$r = a + N^{(-)}, \quad r = a + N^{(+)} \tag{2.9a, b}$$

then the appropriate definition for $h(\theta)$ is

$$h = \frac{1}{a}\int_{a+N^{(-)}}^{a+N^{(+)}} r\,\mathrm{d}r = (N^{(+)} - N^{(-)})\left[1 + \frac{N^{(+)} + N^{(-)}}{2a}\right] \tag{2.10}$$

(Smith 1990, equation (3.6*a*)). For concentric cylinders the quadratic correction in the formula (2.9) vanishes if we define a as the mean of the outer and inner radii. We remark that the analyses of Muller & Bittleston (1991) and of Sankarasubramanian

& Gill (1971) for the non-rotating case are applicable for gaps of arbitrary width. These authors use the outer radius as the reference length. So, care must be taken when comparing their numerical results with the analytic expressions derived in the present paper (see §8).

3. Solutions for the flow around and along the narrow gap

It is a straightforward application of lubrication theory (Reynolds 1886; Schlichting 1955, section 6*c*) to solve (2.1), (2.2*a*, *b*) for the flow. Angle brackets will be used to denote θ- averages, e.g.

$$\langle h \rangle = \frac{1}{2\pi}\int_0^{2\pi} h\,\mathrm{d}\theta. \tag{3.1}$$

Over bars denote cross-sectional average values, e.g.

$$\bar{v} = \frac{\langle hv \rangle}{\langle h \rangle} = \frac{1}{2\pi\langle h \rangle}\int_0^{2\pi} vh\,\mathrm{d}\theta. \tag{3.2}$$

The mass conservation equation implies that the local velocity $v(\theta)$ around the axle scales inversely as the local gap width:

$$v = \frac{\bar{v}\langle h \rangle}{h}. \tag{3.3}$$

This relationship enables us to rewrite the transverse momentum equation (2.2*a*):

$$\frac{\bar{v}\langle h \rangle}{h^3} - \frac{\Omega a}{2h^2} = -\frac{1}{12a\rho\nu}\partial_\theta p. \tag{3.4}$$

Periodicity of the excess pressure p upon going around the axle, enables us to solve (3.4) for the mean circulation velocity $\bar{v}$:

$$\bar{v} = \frac{\Omega a\langle h^{-2} \rangle}{2\langle h \rangle\langle h^{-3} \rangle}. \tag{3.5}$$

In particular, when the gap width does not vary with θ, then v has the constant value $\frac{1}{2}\Omega a$. For variable $h(\theta)$ the resistance to the circulation is dominated by regions of small gap width (cf. Schlichting 1955, equation (6.20)).

Since neither the velocity $v(\theta)$ nor the gap width $h(\theta)$ varies with z, it follows from the transverse momentum equation (2.2*a*) that

$$\partial_\theta \partial_z p = 0. \tag{3.6}$$

Hence, the pressure gradient $\partial_z p$ in the longitudinal momentum equation (2.2*b*) does not vary with θ. Consequently, the local axial velocity $w(\theta)$ is proportional to the square of the local gap width:

$$w = h^2\left(\frac{-\partial_z p}{12\nu\rho}\right). \tag{3.7}$$

In terms of the mean axial velocity $\bar{w}$, the solutions for $w(\theta)$ and for $\partial_z p$ can be written

$$w = h^2\frac{\langle h \rangle\bar{w}}{\langle h^3 \rangle}, \quad \partial_z p = -12\frac{\langle h \rangle}{\langle h^3 \rangle}\rho\nu\bar{w}. \tag{3.8a, b}$$

When the gap width does not vary with θ, then w has the constant value $\bar{w}$. In contrast to the transverse velocity (3.5), it is regions of wide gap that dominate the resistance to the axial flow.

4. Taylor limit

As the solute travels along the axle, the longitudinal gradients of concentration become smaller (Muller & Bittleston 1991), and the mean advection velocity for the solute can be approximated more and more closely by the mean advection velocity $\bar{w}$ for the flow. The timescale on which mixing takes place around the circumference can be estimated as

$$t_{\text{circ}} = \frac{a^2}{K+D_{\theta\theta}}. \tag{4.1}$$

This is much longer than the time (2.6) for mixing across the gap. Indeed, in many circumstances there is insufficient time/length of shaft for there to be complete mixing around the axle (Long 1990; Muller & Bittleston 1991). The shear contribution $D_{\theta\theta}$ to the mixing helps to reduce the mixing time.

To focus our attention upon the even slower longitudinal dispersion, we introduce a small parameter δ, and we define a moving stretched coordinate system

$$\zeta = \delta(z-\bar{w}t), \quad \tau = \delta^2 t. \tag{4.2a, b}$$

These scalings merely formalize the heuristic derivation given by Taylor (1953). The rescaled version of the two-dimensional shear dispersion equation (2.3) is

$$\delta^2 h\,\partial_\tau c + \frac{\langle h\rangle\bar{v}}{a}\partial_\theta c + \delta h(w-\bar{w})\,\partial_\zeta c = \frac{1}{a^2}\partial_\theta[h(K+D_{\theta\theta})\,\partial_\theta c] + \frac{\delta}{a}\partial_\theta(hD_{\theta z}\,\partial_\zeta c)$$

$$+\frac{\delta}{a}\partial_\zeta(hD_{z\theta}\,\partial_\theta c) + \delta^2\,\partial_\zeta[h(K+D_{zz})\,\partial_\zeta c]. \tag{4.3}$$

In the Taylor limit (Taylor 1953), for which the rescaling (4.2*a*, *b*) is appropriate, the solute concentration becomes nearly well-mixed around the axle. So, we write the concentration as a small perturbation about $\bar{c}$:

$$c = \bar{c} + \delta c'. \tag{4.4}$$

To leading order in δ the equations satisfied by $\bar{c}$ and by c' are

$$\langle h\rangle\,\partial_\tau \bar{c} + \partial_\zeta\langle h(w-\bar{w})\,c'\rangle = \frac{1}{a}\partial_\zeta\langle hD_{z\theta}\,\partial_\theta c'\rangle + \langle h(K+D_{zz})\rangle\,\partial_\zeta^2\bar{c}, \tag{4.5a}$$

$$\frac{\langle h\rangle\bar{v}}{a}\partial_\theta c' - \frac{1}{a^2}\partial_\theta[h(K+D_{zz})\,\partial_\theta c'] = h(\bar{w}-w)\,\partial_\zeta\bar{c} + \frac{1}{a}\partial_\theta(hD_{\theta z})\,\partial_\zeta\bar{c}. \tag{4.5b}$$

The new features not present in the work of Taylor (1953, equation (19)) or of Sankarasubramanian & Gill (1971, equation (30)) are the circulation velocity $\bar{v}$ and the off-diagonal shear dispersion $D_{\theta z} = D_{z\theta}$.

5. Longitudinal distortion of constant-concentration surfaces

To emphasize the fact that at leading order c' is proportional to $\partial_\zeta \bar{c}$ we write

$$c' = -f(\theta)\,\partial_\zeta\bar{c} \quad \text{with} \quad \bar{f} = 0, \tag{5.1}$$

where $f(\theta)$ satisfies the circumferential advection–diffusion equation

$$\frac{\langle h\rangle\bar{v}}{a}\partial_\theta f - \frac{1}{a^2}\partial_\theta[h(K+D_{\theta\theta})\,\partial_\theta f] = h(w-\bar{w}) - \frac{1}{a}\partial_\theta(hD_{\theta z}). \tag{5.2}$$

Physically, the function $f(\theta)$ can be interpreted as the longitudinal distortion of a constant-concentration surface. When the gap width h does not vary with θ, the right-hand side forcing in (5.2) vanishes, and f is identically zero. In terms of $f(\theta)$ the longitudinal dispersion equation (4.5a) can be written

$$\partial_\tau \bar{c} = (K+\overline{D_{zz}}+\overline{(w-\bar{w})f}-\overline{D_{z\theta}\partial_\theta f/a})\,\partial_\xi^2 \bar{c}. \tag{5.3}$$

Equation (5.2) can be converted to constant-coefficient form by the change of angular coordinate

$$\psi = \frac{K+\hat{D}}{\langle h^{-1}\rangle}\int_0^\theta \frac{\mathrm{d}\theta'}{h(K+D_{\theta\theta})}, \tag{5.4a}$$

with
$$K+\hat{D} = \frac{\langle h^{-1}\rangle}{\langle h^{-1}(K+D_{\theta\theta})^{-1}\rangle}. \tag{5.4b}$$

Here $\hat{D}$ can be interpreted as being the effective mean value of the circumferential shear dispersion $D_{\theta\theta}$. The simplified equation satisfied by $f(\psi)$ is

$$\frac{\bar{v}}{a}\partial_\psi f-\frac{1}{a^2}\frac{K+\hat{D}}{\langle h\rangle\langle h^{-1}\rangle}\partial_\psi^2 f = \frac{h^2\langle h^{-1}\rangle(K+D_{\theta\theta})}{\langle h\rangle(K+\hat{D})}(w-\bar{w})-\frac{1}{a\langle h\rangle}\partial_\psi(hD_{\theta z}). \tag{5.5}$$

For later use we define the dimensionless parameter

$$\mu = \frac{1}{\langle h\rangle\langle h^{-1}\rangle}. \tag{5.6}$$

To take full advantage of the constant-coefficient left-hand side of (5.5), we pose Fourier series for the right-hand-side forcing:

$$\frac{h^2(K+D_{\theta\theta})}{\langle h\rangle^2\mu(K+\hat{D})}(w-\bar{w}) = \sum_{n=1}^{\infty} w_n^{(c)}\cos n\psi+\sum_{n=1}^{\infty} w_n^{(s)}\sin n\psi, \tag{5.7a}$$

$$\frac{h}{\langle h\rangle}D_{\theta z} = d_0+\sum_{n=1}^{\infty} d_n^{(c)}\cos n\psi+\sum_{n=1}^{\infty} d_n^{(s)}\sin n\psi. \tag{5.7b}$$

The Fourier coefficients can be defined by θ-integrals:

$$w_n^{(c)} = \frac{1}{\pi\langle h\rangle}\int_0^{2\pi} h(w-\bar{w})\cos n\psi\,\mathrm{d}\theta = 2\overline{(w-\bar{w})\cos n\psi}, \tag{5.8a}$$

$$w_n^{(s)} = \frac{1}{\pi\langle h\rangle}\int_0^{2\pi} h(w-\bar{w})\sin n\psi\,\mathrm{d}\theta = 2\overline{(w-\bar{w})\sin n\psi}, \tag{5.8b}$$

$$d_0 = \frac{\mu(K+\hat{D})}{2\pi}\int_0^{2\pi}\frac{D_{\theta z}\,\mathrm{d}\theta}{(K+D_{\theta\theta})}, \tag{5.8c}$$

$$d_n^{(c)} = \mu\frac{(K+\hat{D})}{\pi}\int_0^{2\pi}\frac{D_{\theta z}}{(K+D_{\theta\theta})}\cos n\psi\,\mathrm{d}\theta, \tag{5.8d}$$

$$d_n^{(s)} = \mu\frac{(K+\hat{D})}{\pi}\int_0^{2\pi}\frac{D_{\theta z}}{(K+D_{\theta\theta})}\sin n\psi\,\mathrm{d}\theta. \tag{5.8e}$$

The corresponding Fourier series for $f(\psi)$ is

$$f(\psi) = \sum_{n=1}^{\infty} f_n^{(c)}(\cos n\psi-\overline{\cos n\psi})+\sum_{n=1}^{\infty} f_n^{(s)}(\sin n\psi-\overline{\sin n\psi}), \tag{5.9a}$$

where

$$f_n^{(c)} = \frac{a[n\mu(K+\hat{D})(aw_n^{(c)} - nd_n^{(s)}) - \bar{v}a(aw_n^{(s)} + nd_n^{(c)})]}{n[n^2\mu^2(K+\hat{D})^2 + \bar{v}^2 a^2]}, \tag{5.9b}$$

$$f_n^{(s)} = \frac{a\{n\mu(K+\hat{D})(aw_n^{(s)} + nd_n^{(c)}) + \bar{v}a(aw_n^{(c)} - nd_n^{(s)})\}}{n[n^2\mu^2(K+\hat{D})^2 + \bar{v}^2 a^2]}. \tag{5.9c}$$

6. Shear dispersion coefficients

From the Fourier representations for $w-\bar{w}$, $D_{\theta z} = D_{z\theta}$, and f we can rewrite the longitudinal dispersion equation (5.3):

$$\partial_\tau \bar{c} = (K + D_{\mathrm{L}} + E)\,\partial_\xi^2 \bar{c} \tag{6.1}$$

Here D_{L} is the effective average value of the longitudinal mixing associated with the shear across the gap:

$$D_{\mathrm{L}} = \bar{D}_{zz} - \frac{1}{2\mu(K+\hat{D})} \sum_{n=1}^{\infty} (d_n^{(c)^2} + d_n^{(s)^2}), \tag{6.2}$$

and E is a longitudinal shear dispersion coefficient associated with shear around the axle:

$$E = \frac{a^2}{2\mu(K+\hat{D})} \sum_{n=1}^{\infty} \frac{[\mu(K+\hat{D})\,w_n^{(c)} - \bar{v}d_n^{(c)}]^2 + [\mu(K+\hat{D})\,w_n^{(s)} - \bar{v}d_n^{(s)}]^2}{n^2\mu^2(K+\hat{D})^2 + \bar{v}^2 a^2}. \tag{6.3}$$

The n^2 factors in the denominator help to accelerate the convergence of this series for E.

To be physically meaningful, a shear dispersion coefficient must be non-negative. All the terms in the summation (6.3) are positive, so E is strictly positive. However, the expression (6.2) for D_{L} needs further investigation. From the Fourier series (5.7*b*) for $D_{\theta z}$, we can rewrite D_{L}:

$$\begin{aligned} D_{\mathrm{L}} &= \frac{d_0^2}{\mu(K+\hat{D})} + \bar{D}_{zz} - \frac{1}{2\pi\langle h\rangle}\int_0^{2\pi} \frac{hD_{\theta z}^2\,\mathrm{d}\theta'}{K + D_{\theta\theta}} \\ &= \frac{d_0^2}{\mu(K+\hat{D})} + \frac{1}{2\pi\langle h\rangle}\int_0^{2\pi} \frac{hKD_{zz}}{(K+D_{\theta\theta})}\,\mathrm{d}\theta' + \frac{1}{2\pi\langle h\rangle}\int_0^{2\pi} \frac{h(D_{\theta\theta}D_{zz} - D_{\theta z}^2)}{K + D_{\theta\theta}}\,\mathrm{d}\theta'. \end{aligned} \tag{6.4}$$

The positive definiteness (2.4*b*) of the shear dispersion tensor permits us to show that the final term in (6.4) is non-negative. So, D_{L} is indeed non-negative. It is a consequence of the residual non-uniformity of concentration around the axle, that the off-diagonal terms $D_{\theta z} = D_{z\theta}$ modify the effective average value of the longitudinal mixing D_{L}.

Increasing either the mean circulation velocity $\bar{v}$ or the circumferential shear dispersion $\hat{D}$ reduces the value of each term in the series (6.3) for E. We can use (3.5) and (2.5*a*) to estimate the dependence of $\bar{v}$ and $\hat{D}$ upon the rotation rate Ω. As Ω increases we can identify three parameter regimes for the size of E depending upon which of K, $\bar{v}a$ and $\hat{D}$ dominates:

$$E = O\left(\frac{a^2\bar{w}^2}{K}\right) \quad \text{for} \quad \Omega < \frac{K}{a^2}, \tag{6.5a}$$

$$E = O\left(\frac{K\bar{w}^2}{\Omega^2 a^2}\right) \quad \text{for} \quad \frac{K}{a^2} < \Omega < \frac{K}{\langle h\rangle^2}, \tag{6.5b}$$

$$E = O\left(\frac{K\bar{w}^2}{\Omega^2\langle h\rangle^2}\right) \quad \text{for} \quad \frac{K}{\langle h\rangle^2} < \Omega. \tag{6.5c}$$

Strictly, the estimate (6.5c) is invalid since in the high-rotation regime the condition (2.7) for mixing across the gap is violated.

For zero rotation Sankarasubramanian & Gill (1971) emphasized that E vastly dominates D_{L}. This dominance continues throughout the low-rotation regime (6.5a). For the middle regime (6.5b) there is a transition of dominance

$$E \sim D_{\mathrm{L}} \quad \text{for} \quad \Omega \sim \frac{K}{a\langle h\rangle}. \tag{6.6}$$

Finally, for rapid rotation E is negligible relative to D_{L} (though the precise estimate (6.5c) is invalid). Hence, in those parameter regimes (6.5a, b) for which E needs to be calculated, we can make the considerable simplification of neglecting both $D_{\theta\theta}$ and $D_{\theta z} = D_{z\theta}$. The simplified formula for E is

$$E(\bar{v}) = \tfrac{1}{2}\mu K a^2 \sum_{n=1}^{\infty} \frac{w_n^{(c)^2} + w_n^{(s)^2}}{n^2\mu^2K^2 + \bar{v}^2a^2}. \tag{6.7}$$

Pedley & Kamm (1988, equation (4.3)) give an equivalent formula with the additional complication of flow oscillations.

For fluid–fluid displacement in an annular region (Long 1990), reduced longitudinal mixing implies a more efficient displacement process (so cleaning can be achieved with less fluid, or there is less residual contamination). Indeed, for boreholes it is becoming standard practice to rotate the cylindrical liner when displacing the fluid in the gap between the liner and the borehole wall (Lindsey & Durham 1984; Arceneaux & Smith 1986). The above formula (6.7) gives a quantitative basis for that practice.

7. One-term approximation

For zero axle rotation (6.7) becomes

$$E(0) = \frac{a^2}{2\mu K} \sum_{n=1}^{\infty} \frac{w_n^{(c)^2} + w_n^{(s)^2}}{n^2}. \tag{7.1}$$

From the Fourier series (5.7a) for $w - \bar{w}$ we can replace the summation by a double integral:

$$E(0) = \frac{a^2\bar{w}^2}{K} I_2, \tag{7.2a}$$

with
$$I_2 = \frac{1}{\langle h\rangle}\left\langle \frac{1}{h}\left\{\int_0^\theta h\left(\frac{w}{\bar{w}} - 1\right)\mathrm{d}\theta' - \frac{1}{\langle h^{-1}\rangle}\left\langle \frac{1}{h}\int_0^\theta h\left(\frac{w}{\bar{w}} - 1\right)\mathrm{d}\theta'\right\rangle\right\}^2\right\rangle. \tag{7.2b}$$

The dimensionless integral I_2 depends only upon the shape of the non-uniform gap around the axle.

For very rapid axle rotation (6.7) has the asymptote

$$E(\bar{v}) \sim \frac{\mu K}{2\bar{v}^2} \sum_{n=1}^{\infty} (w_n^{(c)^2} + w_n^{(s)^2}) \quad \text{for large} \quad \bar{v}. \tag{7.3}$$

Again, the Fourier series (5.7a) for $w - \bar{w}$ enables us to replace the summation by an integral

$$E(\bar{v}) \sim \frac{K\bar{w}^2}{\bar{v}^2} I_1 \quad \text{for large} \quad \bar{v}, \tag{7.4a}$$

with
$$I_1 = \left\langle \left(\frac{h}{\langle h\rangle}\right)^3 \left(\frac{w}{\bar{w}} - 1\right)^2 \right\rangle. \tag{7.4b}$$

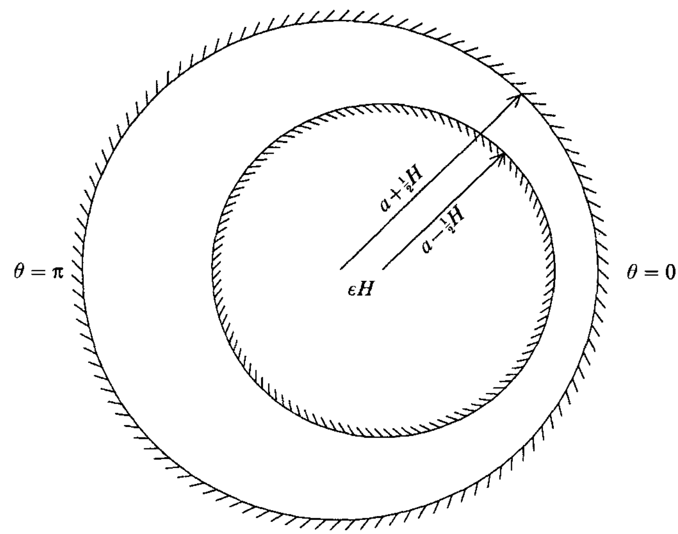

FIGURE 2. An off-centre axle in a cylindrical shaft.

As before, the dimensionless integral I_1 depends only upon the shape of non-uniform gap.

A one-term approximation for $E(\bar{v})$ which interpolates between the limiting forms (7.2a), (7.4a) is

$$E(\bar{v}) = \frac{E(0)}{1+(\bar{v}/v_{\mathrm{R}})^2}, \tag{7.5a}$$

with

$$v_{\mathrm{R}} = \frac{K}{a}\left(\frac{I_1}{I_2}\right)^{\frac{1}{2}}. \tag{7.5b}$$

Here v_{R} is reference value for the rotation velocity, beyond which the shear dispersion contribution $E(\bar{v})$ decreases rapidly.

For arbitrary-shaped gaps $h(\theta)$, it is a straightforward computational task to use the approximation (7.5a). First, we use (3.5) and (3.8a) to evaluate the mean transverse velocity $\bar{v}$ and the shape of the longitudinal velocity profile $w(\theta)/\bar{w}$. Next, we evaluate the dimensionless integrals (7.2b), (7.4b). Finally, we evaluate $E(0)$ from (7.2a) and the rotation-modified shear dispersion coefficient from (7.5a, b). A test of the one-term approximation is given in the next section.

8. Non-concentric cylinders

As an illustrative example, we assume that the shaft is cylindrical with a nominal clearance H from the axle. If there is a displacement ϵH between the two centres, then the non-uniform gap width $h(\theta)$ can be represented (see figure 2):

$$h(\theta) = H(1-\epsilon\cos\theta), \quad -1 < \epsilon < 1. \tag{8.1}$$

The angle θ is measured around the axle from the region of narrowest gap. From (3.5) we can evaluate the mean circulation velocity

$$\bar{v} = \Omega a\frac{1-\epsilon^2}{2+\epsilon^2}, \tag{8.2}$$

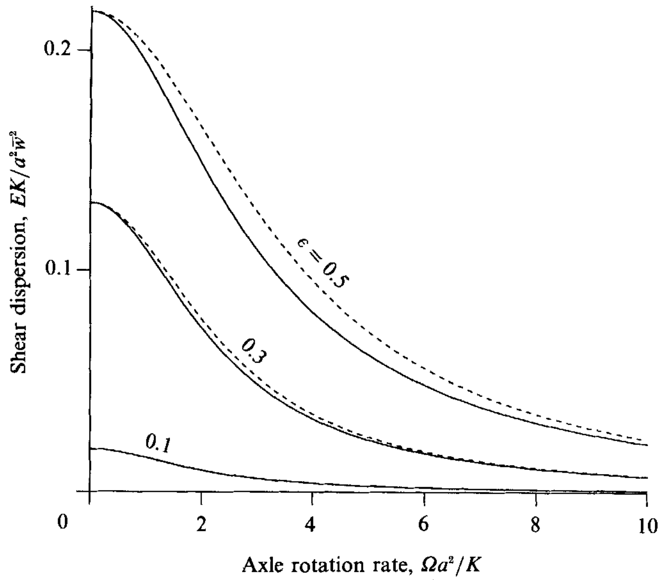

FIGURE 3. Comparison between the exact (——) and one-term approximation (----) to the non-dimensional shear dispersion coefficient E^* for eccentric annuli with fractional offsets $\epsilon = 0.1, 0.3, 0.5$.

and from (3.8a) we can determine the shape of the longitudinal velocity profile

$$\frac{w(\theta)}{\bar{w}} - 1 = \frac{\epsilon}{1+\frac{3}{2}\epsilon^2}(-2\cos\theta - \tfrac{3}{2}\epsilon + \epsilon\cos^2\theta). \tag{8.3}$$

For small eccentricity ϵ the span of velocities increases with ϵ, reaching a maximum at $\epsilon = 0.82$.

The parameter μ defined in (5.6) has the value

$$\mu = (1-\epsilon^2)^{\frac{1}{2}}, \tag{8.4}$$

and the modified angular coordinate ψ is given by the formula

$$\psi = 2\arctan\left[\left(\frac{1+\epsilon}{1-\epsilon}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}\theta\right]. \tag{8.5}$$

For a given value of ϵ the Fourier coefficients $w_n^{(c)}$ can be evaluated numerically via the integrals (5.8a). The sine coefficients $w_n^{(s)}$ are zero by symmetry. Finally, the summation (6.7) yields the axial velocity contribution E to the longitudinal shear dispersion. The continuous curves in figure 3 show the non-dimensional dispersion coefficient

$$E^* = \frac{EK}{a^2\bar{w}^2} \tag{8.6}$$

as a function of the non-dimensional rotation rate

$$\Omega^* = \frac{\Omega a^2}{K}. \tag{8.7}$$

The one-term approximation described in the previous section has the advantage that all the calculations can be performed analytically:

$$I_1 = \frac{\epsilon^2}{(1+\frac{3}{2}\epsilon^2)^2}(2+\tfrac{9}{8}\epsilon^2-\tfrac{1}{16}\epsilon^4), \tag{8.8a}$$

$$I_2 = \frac{\epsilon^2}{(1+\frac{3}{2}\epsilon^2)^2}\left\{2-\frac{31}{36}\epsilon^2-\frac{25}{72}\left[1-\frac{2(1-\epsilon^2)^{\frac{3}{2}}}{1+(1-\epsilon^2)^{\frac{1}{2}}}\right]\right\}. \tag{8.8b}$$

We remark that I_2 has a maximum value 0.23 at $\epsilon = 0.6$. The dotted curves in figure 3 show the efficacy of the one-term approximation (7.5*a*, *b*).

If $D_{\theta z} = D_{z\theta}$ is negligible, then from (2.5*c*) we can evaluate the non-dimensional counterpart to D_{L}:

$$D^*_{\mathrm{L}} = \bar{D}^*_{zz} = \left(\frac{H}{a}\right)^2 \frac{1}{210(1+\frac{3}{2}\epsilon^2)^2}(1+\tfrac{21}{2}\,\epsilon^2+\tfrac{105}{8}\epsilon^4+\tfrac{35}{16}\,\epsilon^6). \tag{8.9}$$

Sankarasubramanian & Gill (1971, figure 5) give numerical results for the non-dimensional shear dispersion coefficient for flow between stationary cylinders with a radius ratio 1.5. These results enable us to test the accuracy of the narrow-gap approximation. As noted in §2, we select our reference radius a to be the mean value of the outer and inner radii. So, the radii are $1.2a$, $0.8a$ with a nominal gap width $H = 0.4a$ (as sketched in figures 1 and 2). When rescaled relative to a (rather than the outer radius) the results given by Sankarasubramanian & Gill (1971, figure 5) become

$$D^*_{\mathrm{L}}+E^* = 8.155\times 10^{-4},\quad 0.01917,\quad 0.2055 \tag{8.10}$$

for eccentricities (fractional offsets) $\epsilon = 0,\ 0.1,\ 0.5$. The narrow-gap formulae (8.8*a*), (8.9) yield the predictions

$$D^*_{\mathrm{L}}+E^* = 7.619\times 10^{-4},\quad 0.02004,\quad 0.2247. \tag{8.11}$$

So, the narrow-gap formulae are correct to within 10%, even though the gap is not very narrow.

Sankarasubramanian & Gill (1971) draw attention to the dramatic, nearly 25-fold increase in the longitudinal dispersion coefficient which occurs when the fractional offset is increased from $\epsilon = 0$ to $\epsilon = 0.1$, even though the eccentricity is barely perceptible by eye. The explicit formulae (8.8*b*), (8.9) reveal that

$$\frac{E^*}{D^*_{\mathrm{L}}} \sim 420\epsilon^2\left(\frac{a}{H}\right)^2 \quad \text{for small} \quad \epsilon. \tag{8.12}$$

So, for narrower gaps the role of eccentricity is even more dominant. In the case $\epsilon = 0.5$, $H = 0.4a$ the shear dispersion coefficient is over 250 times that for the concentric case, and the ratio would be even more extreme if the gap were narrower.

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